

Cork Institute of Technology

Special Mathematics Examination for Engineering Degree Entry

June 2008

Time: 2 hours and 30 minutes

PAPER 1 (300 marks)

Attempt **SIX** questions.
Each question carries 50 marks.

WARNING: Marks will be lost if all necessary work is not clearly shown.

**Answers should include the appropriate units of measurement,
where relevant.**

1. (a) Solve the simultaneous equations

$$\frac{1}{x} - 2 = 0$$
$$x^2 + xy = 1.$$

(b) Let $f(x) = x^3 + kx^2 + x + k$.

(i) Given that $x - 3$ is a factor of $f(x)$, find k .

(ii) Hence, solve the equation $f(x) = 0$.

(c) Given that $a > 0$, $b > 0$ and $a, b \in \mathbf{R}$, show that

(i) $\frac{a}{b} + \frac{b}{a} \geq 2$

(ii) $(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4.$

2. (a) Solve for x : $|x - 3| < 5$, where $x \in \mathbf{R}$.

(b) (i) Write $\frac{1}{2\sqrt{12}} - \frac{1}{5\sqrt{3}}$ in the form $\frac{\sqrt{3}}{k}$, where $k \in \mathbf{N}$.

(ii) If $x = \frac{2t}{1+t^2}$ and $y = \frac{1-t^2}{1+t^2}$, find the value of $x^2 + y^2$.

(c) Let $f(x) = x^2 + (2a+1)x + (a^2 + a) = 0$.

(i) Show that the roots of the equation $f(x) = 0$ are real.

(ii) Express these roots in terms of a .

3. (a) Evaluate $\begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ -3 \end{pmatrix}$.

(b) (i) Given that

$$(1-x) + (x+2y)i = (2x-5y) + (3+y)i$$

find the value of x and the value of y , where $x, y \in \mathbf{R}$ and $i^2 = -1$.

(ii) Find the complex number z , given that

$$\frac{2z}{z+1} = 1+i.$$

Write your answer in its simplest form.

(c) (i) Prove by induction that

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta, \quad n \in \mathbf{N}_0.$$

(ii) If $z = \cos \theta + i \sin \theta$, show that

$$z^n + \frac{1}{z^n} = 2 \cos n\theta.$$

4. (a) The first n terms of an arithmetic sequence are 2, 6, ... $(4n-2)$.
Find which term of the sequence has the value of 5202.

(b) (i) The sum to infinity, S_∞ , of a geometric series is 8.
The first term of the series is 4.
Find the common ratio.

(ii) Show that $S_\infty - S_{20} = 2^{-17}$, where S_{20} is the sum of the first 20 terms of the series.

(c) (i) Evaluate $\sum_{n=1}^{40} n^2$.

(ii) Hence, or otherwise, find the value $1^2 + 3^2 + 5^2 + \dots + 37^2 + 39^2$.

5. (a) Solve for x : $5^{2x} = 125^{x-1}$.

Verify your answer.

(b) In the expansion of $\left(x^2 - \frac{1}{x}\right)^{12}$ find

(i) the general term of the expansion

(ii) the coefficient of the term with x^3 .

(c) The n th term of a series is given by $\frac{n}{3^n}$.

(i) Write the first three terms of the series.

(ii) Find an expression for S_n , the sum of the first n terms of the series.

(iii) Hence, find the sum to infinity of the series.

6. (a) Differentiate $\frac{1-x^2}{2+x}$ with respect to x .

(b) (i) Differentiate $x^2 \log x$ with respect to x .

(ii) Let $y = \cos^4 x$.

Find $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$.

(c) Let $y = \sqrt{1+x^2}$.

(i) Find $\frac{dy}{dx}$.

(ii) Show that $y^3 \frac{d^2y}{dx^2} = 1$.

7. (a) Differentiate $\cos(4x - 2)$ with respect to x .

(b) (i) Given that

$$2xy^2 = x^2 + 1,$$

find the value of $\frac{dy}{dx}$ at the point $(1, 1)$.

(ii) Show that $2x^3 - 6x + 2 = 0$ has a root between 0 and 1.

Taking $x_1 = 0$ as the first approximation, use the Newton-Rhapson method to calculate the second approximation of this root.

(c) A curve is defined by the parametric equations

$$x = 1 + e^{-t}, \quad y = t^2 + 2e^t.$$

(i) Show that $\frac{dy}{dx} = -2e^t(t + e^t)$.

(ii) Hence, find the equation of the tangent to the curve at the point $x = 2$.

8. (a) Find (i) $\int (5x^2 - 4) dx$ (ii) $\int \sin 4x dx$.

(b) (i) Evaluate $\int_1^{\sqrt{2}} xe^{x^2} dx$.

(ii) Evaluate $\int_0^1 \frac{x^2 - x}{x + 3} dx$.

(c) Find the area of the region enclosed by the curve $f(x) = x(8 - x)$ and the line $g(x) = x$.