Cork Institute of Technology

Special Mathematics Examination for Engineering Degree Entry

June 2008

Time: 2 hours and 30 minutes

PAPER 2 (300 marks)

Attempt **FIVE** questions from Section **A** and **ONE** question from Section **B**.
Each question carries 50 marks.

WARNING: **Marks will be lost if all necessary work is not clearly shown.**

Answers should include the appropriate units of measurement, where relevant.

SECTION A

Answer FIVE questions from this section

1. (a) The point $(p, 3p)$ is on the circle $x^2 + y^2 = 40$.
Find the value of p, where $p > 0$.

(b) The circle C passes through the points $(-2, 4)$ and $(0, 2)$.
The x-axis is a tangent to C.
Find the two possible equations of C.

(c) The equation of a circle is $x^2 + y^2 - 2x - 4y - 20 = 0$.
The line $2y = x + k$ contains a chord of the circle of length $4\sqrt{5}$.
Find k.

2. (a) Given that $2\vec{i} - 3\vec{j} + \vec{k} = -3(\vec{i} + 6\vec{j})$, find $\vec{k}$ in terms of $\vec{i}$ and $\vec{j}$.

(b) $\vec{a} = \vec{i} - \vec{j}$ and $\vec{b} = 4\vec{i} + 3\vec{j}$.

(i) Given that θ is the angle between $\vec{a}$ and $\vec{b}$, show that $\theta = \cos^{-1}\dfrac{1}{5\sqrt{2}}$.

(ii) Find the area of the triangle *oab*.

(c) $\vec{r} = 3\vec{i} + 4\vec{j}$ and $\vec{s} = 3\vec{i} - 4\vec{j}$.

Express

(i) $\vec{u} = \dfrac{\vec{r}}{|\vec{r}|}$ in terms of $\vec{i}$ and $\vec{j}$.

(ii) $\vec{v} = \vec{s} - \left(\vec{s}.\vec{u}\right)\vec{u}$ in terms of $\vec{i}$ and $\vec{j}$.

(iii) Prove that $\vec{v} \perp \vec{u}$.

3. (a) Find the area of the triangle with vertices $(-6, 0)$, $(4, 0)$ and $(-2, 8)$.

(b) **(i)** Prove that the angle θ between the two lines with slopes m_1 and m_2 is given by

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2}.$$

(ii) Find the slopes of the lines which each make an angle of $45°$ with the line $3x - y = 0$.

(c) **(i)** Find the value of λ and the value of μ for which

$$8x + 25y - 41 = \lambda(2x + 5y + 1) + \mu(x + 3y - 4).$$

(ii) Hence, or otherwise, explain why the lines

$$2x + 5y + 1 = 0, \quad x + 3y - 4 = 0 \quad \text{and} \quad 8x + 25y - 41 = 0$$

are concurrent.

4. (a) Evaluate $\underset{x\to 0}{Limit}\dfrac{\sin 3x}{\sin 2x}$.

(b) **(i)** Prove that $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$

(ii) Using $\tan A = \sqrt{3}$, $0 \le A \le \pi$, verify the above result.

(c) In the triangle pqr, t is the midpoint of $[qr]$.

If $|pq| = x$, $|pr| = y$, $|pt| = h$ and $|qt| = k$, show that $x^2 + y^2 = 2h^2 + 2k^2$.

5. (a) Write $\cos 5A \sin 6A$ as a sum of two sines.

(b) Using $\cos 2A = \cos^2 A - \sin^2 A$, prove that $\cos 2A = 1 - 2\sin^2 A$.

Hence, or otherwise, solve the equation

$$\cos 2A = \sin A, \qquad 0 \le A \le 2\pi.$$

(c) **(i)** Show that $\tan\left(\dfrac{\pi}{4} - 2A\right) = \dfrac{1 - \tan 2A}{1 + \tan 2A}$

(ii) Hence, or otherwise, express $\tan\left(\dfrac{\pi}{4} - 2A\right)$ in terms of $\tan A$.

6. (a) A fair die is thrown twice.

What is the probability of throwing a "6" once in the two throws?

(b) Solve the difference equation

$$u_{n+2} + u_{n+1} = 12u_n, \quad n \geq 0$$

given that $u_0 = 8$ and $u_1 = 3$.

Show that $u_2 + u_3 = 36$.

(c) There are 8 boys and 6 girls in a class. One boy has one sister in the class.
Four students are selected from the class.
Find the number of different selections that can be made if

(i) there are no restrictions

(ii) the selection must consist of two boys and two girls

(iii) the selection must consist of two boys and two girls but the boy and his sister cannot be together in the same selection.

7. (a) A letter is chosen at random from the word STATISTICS.
What is the probability that the letter chosen is

(i) the letter S

(ii) not the letter T or the letter C?

(b) **(i)** Simplify $\dfrac{(n+1)!(n-1)!}{(n!)^2}$.

(ii) Solve for n: $\dfrac{5}{8}\dbinom{n}{3} = \dbinom{n-1}{4}$.

(c) $\bar{x}$ and σ are, respectively, the mean and standard deviation of the set $\{x_1, x_2, x_3, \ldots x_n\}$.

Find the mean and standard deviation of the set $\{kx_1, kx_2, kx_3, \ldots kx_n\}$.

SECTION B

Answer ONE question from this section.

8. (a) Use integration by parts to find $\int xe^{-x}dx$.

(b) $f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \ldots$ is the Maclaurin series.

Find the first four terms of the Maclaurin series for $f(x) = \sin x$.

Write down the general term and use the Ratio Test to show that the series converges for all $x \in \mathbf{R}$.

(c) A piece of wire 10 metres long is cut into two pieces.
One piece has length x metres and is shaped into a circle.
The remaining piece is shaped into a square.

(i) Write an expression for the combined area of the circle and square.

(ii) Find the value of x, in terms of π, if the combined area is a minimum.

9. (a) A person is selected at random from a group of 4 men and 6 women.
A second person is then selected from the group.
Find the probability that the second person selected is a woman.

(b) The probability of a particular team winning whenever it plays is $\frac{2}{3}$.
In a series of six games, find the probability that

(i) the team wins exactly three games

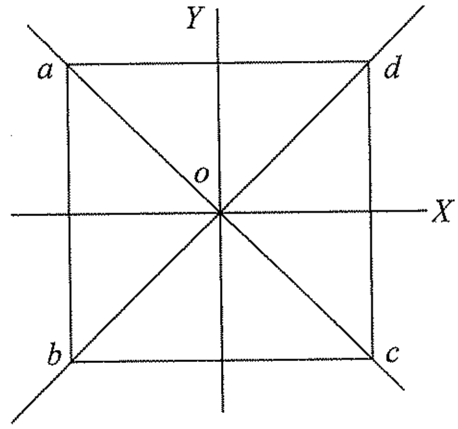
(ii) the team wins more than four games.

(c) A die has three faces coloured red and three faces coloured black.
The die is thrown 1024 times and red shows on 560 of the throws.
At the 5% level of significance, does this result suggest that the die is biased?

10. (a) The binary operation $*$ is defined by $x * y = \frac{x+y}{2}$.

Show that this operation is not associative.

(b) The square *abcd* has eight symmetries.



(i) Name each symmetry and write it using permutation notation e.g.

$$R_{90} = \begin{pmatrix} a & b & c & d \\ b & c & d & a \end{pmatrix} \text{ or } S_{ac} = \begin{pmatrix} a & b & c & d \\ a & d & c & b \end{pmatrix}.$$

(ii) Show that the composition

S_{ac} after R_{90}

is an element of the set of symmetries and name this element.

(c) Prove that any infinite cyclic group is isomorphic to the group **Z**, +.

11. (a) The polar of a point *p* with respect to the circle $x^2 + y^2 = 9$ is $2x - 5y = 27$.
Given that the polar of the point (x_1, y_1) with respect to the circle $x^2 + y^2 = r^2$ is $xx_1 + yy_1 = r^2$, find the co-ordinates of *p*.

(b) Show that the ratios of the lengths of line segments in parallel lines are invariant under every affine transformation.

(c) *f* is a similarity transformation.

The image of the line segment $[pq]$ under *f* is the line segment $[p'q']$.

If the line *M* is the perpendicular bisector of $[pq]$, prove that *f*(*M*) is the perpendicular bisector of $[p'q']$.