# Cork Institute of Technology

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## Special Mathematics Examination for Engineering Degree Entry

**June 2009**

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**Time: 2 hours and 30 minutes**

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**PAPER 1 (300 marks)**

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Attempt **SIX** questions.
Each question carries 50 marks.

WARNING: **Marks will be lost if all necessary work is not clearly shown.**

**Answers should include the appropriate units of measurement, where relevant.**

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1. (a) Simplify fully

$$\frac{x}{x-y} - \frac{y^2}{x^2-xy}.$$

(b) (i) Solve $\sqrt{x} - \frac{9}{\sqrt{x}} = 8$, where $x > 0$.

(ii) Find the range of values of $x \in \mathbf{R}$ for which

$$\frac{3x+1}{x-2} > 2, \quad x \neq 2.$$

(c) (i) Find the range of values of $k$ for which the roots of the equation

$$kx^2 + 8x + k = 0$$

are real.

(ii) For what values of $k$ are the roots of the equation

$$kx^2 + 8x + k = x^2 + 1$$

equal?

2. (a) Let $f(x) = \frac{2^x + 1}{2^{2x}}$.

Find the value of $f(-1)$.

(b) The roots of the equation $2x^2 - 11x + 12 = 0$ are $\alpha$ and $\beta$.

(i) Find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$.

(ii) Find the quadratic equation whose roots are $\alpha^2$ and $\beta^2$.

(c) Let $f(x) = 2x^3 + hx^2 - 13x + k$.

(i) Given that $x^2 - x - 6$ is a factor of $f(x)$, find the value of $h$ and the value of $k$.

(ii) Find the roots of $f(x) = 0$.

**3. (a)** Solve for $x$ and $y$: $\begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

**(b)** Let $z = 5 - 5i$, where $i^2 = -1$.

**(i)** Plot $z$ on an Argand diagram.

**(ii)** Show that $(2+i)(1-3i) = z$.

**(iii)** Given that

$$\frac{\sqrt{2}(\cos\theta + i\sin\theta)}{1-3i} = \frac{2+i}{5},$$

find $\theta$, where $0 \le \theta \le 2\pi$.

**(c)** Let $z = 1 + \sqrt{3}i$.

**(i)** Express $z$ in the form $r(\cos\theta + i\sin\theta)$.

**(ii)** Show that $z^{13} = (2^{12})z$.

**(iii)** Find the square roots of $z$.
Write your answers in the form $x + yi$, where $x, y \in \mathbf{R}$.

**4. (a)** Find the sum to infinity of the geometric series $\frac{1}{2} + \frac{1}{6} + \ldots$

**(b)** A series is defined by

$$u_{n+1} = \frac{u_n + 1}{u_n - 1}, \quad u_1 = \frac{1}{4}.$$

**(i)** Find the first three terms of the series.

**(ii)** Find the sum of the first 18 terms of the series.

**(iii)** The sum of the first $k$ terms of the series is less than $-25$. Find the least value of $k$.

**(c)** The first four terms of an arithmetic sequence are:

$$u_1 = 0, \quad u_2 = p - q, \quad u_3 = 4p + q + 5, \quad u_4 = -5q.$$

**(i)** Find the value of $p$ and the value of $q$.

**(ii)** Find $u_5$, the fifth term of the sequence.

**(iii)** Given that $u_k = 575$, find the value of $k$.

**5. (a)** Find the value of $k$ which satisfies the equation

$$\log_e 9 + \log_e 729 = k\log_e 3.$$

**(b)** **(i)** Solve the equation

$$2^{x^2} = \frac{2^{4x}}{8}.$$

**(ii)** Solve the equation

$$2x + \sqrt{x^2 + 9} = 4x - 3.$$

**(c)** Prove by induction that for $n \geq 1$

$$1^2 + 2^2 + 3^2 + \ldots n^2 = \frac{n(n+1)(2n+1)}{6}.$$

**6. (a)** Differentiate $(x^2 + 3x)^5$ with respect to $x$.

**(b)** **(i)** Differentiate $\frac{1}{x}$ with respect to $x$ from first principles.

**(ii)** The parametric equations of a curve are

$$x = \sin t - 2\cos t \quad \text{and} \quad y = 2\sin t - \cos 2t.$$

Find $\frac{dy}{dx}$ when $t = 0$.

**(c)** Let $f(x) = (x-1)(2x+1)^2$.

**(i)** Find the co-ordinates of the points at which the curve $y = f(x)$ cuts the $x$-axis and the $y$-axis.

**(ii)** Find the co-ordinates of the turning points of the curve $y = f(x)$ and state which is a local maximum and which is a local minimum.

**(iii)** Draw a sketch of the curve $y = f(x)$.

7. (a) Taking 2 as the first approximation of a root of $x^3 - 2x = 0$, use the Newton-Rhapson method to calculate the second approximation of this root.

(b) (i) Differentiate $\sqrt{\frac{4x}{x+2}}$ with respect to $x$ and simplify your answer.

(ii) Let $y = \frac{1}{x^3}\log_e x$.

Find the value of $\frac{dy}{dx}$ at $x = \sqrt{e}$.

(c) The equation of a curve is

$$2x^2 + y^2 + 3x - 6y + 9 = 0.$$

(i) Find $\frac{dy}{dx}$ in terms of $x$ and $y$.

(ii) $(h, k)$ is a point on the curve at which $k = -4h$.
Find the slope of the tangent to the curve at $(h, k)$.

8. (a) Find (i) $\int (1-x)^2\,dx$ (ii) $\int e^{6x}\,dx$.

(b) (i) Evaluate $\int_0^{\frac{\pi}{8}} \sin\left(2x + \frac{\pi}{4}\right)dx$.

(ii) Evaluate $\int_0^1 \frac{x}{x^2+1}\,dx$.

(c) The diagram shows the curve $y = x^2$ and the line $2x + y = 0$.

Find the area of the shaded region enclosed by the curve and the line.