Cork Institute of Technology

Special Mathematics Examination for Engineering Degree Entry

June 2009

Time: 2 hours and 30 minutes

PAPER 2 (300 marks)

Attempt **FIVE** questions from Section A and **ONE** question from Section **B**.
Each question carries 50 marks.

WARNING: **Marks will be lost if all necessary work is not clearly shown.**

Answers should include the appropriate units of measurement, where relevant.

SECTION A
Answer FIVE questions from this section

1. (a) C is the circle $x^2 + y^2 - 10x + 8y + 16 = 0$. $t(1, -1)$ is a point.

(i) Write down the centre and radius of C.

(ii) Verify that t is a point on C.

(iii) Show that the y-axis is a tangent to C.

(iv) C intersects the x-axis at p and at q. Find $|pq|$.

(v) Find the equation of the tangent to C at the point t.

(iv) Write down the equation of the circle with centre (0, 0) and passing through the centre of C.

(b) S_1 is the circle $x^2 + y^2 + 6y + 8 = 0$ and S_2 is the circle $x^2 + y^2 - 12x - 10y - 60 = 0$.

(i) Show that the two circles touch each other.

(ii) Find the equation of the tangent to the circles at the point of contact.

(iii) Find the equation of the smallest circle passing through the centres of S_1 and S_2.

2. (a) Let $\vec{x} = 5\vec{i} + 12\vec{j}$.

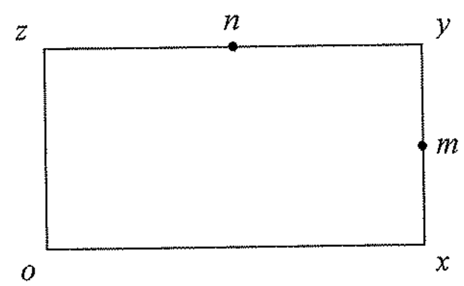
(i) Find $\vec{x}^{\perp}$ in terms of $\vec{i}$ and $\vec{j}$.

(ii) Find the unit vector perpendicular to $\vec{x}$.

(b) $oxyz$ is a rectangle, where o is the origin.
m is the midpoint of $[xy]$ and n is the midpoint of $[zy]$.

(i) Express $\vec{mn}$ in terms of $\vec{x}$ and $\vec{z}$.

(ii) Deduce that $|xz| = 2|mn|$.



(c) $owvu$ is a parallelogram, where o is the origin.
$\vec{v} = 3\vec{i} + 4\vec{j}$ and $\vec{uv} = 5\vec{i} + \vec{j}$.

(i) Express $\vec{w}$ in terms of $\vec{i}$ and $\vec{j}$.

(ii) Find $\vec{u}$ in terms of $\vec{i}$ and $\vec{j}$.

(iii) Find $|\angle vow|$, correct to the nearest degree.

(iv) $\vec{s}$ is the vector $h\vec{i} + k\vec{j}$. $\vec{s} \perp \vec{u}$ and $\vec{s} = t\vec{u} - \vec{v}$ where $t \in \mathbf{R}$.
Find the value of h and the value of k.

3. (a) The parametric equations

$$x = 2t + 1 \text{ and } y = 4t - 6$$

represent a line, where $t \in \mathbf{R}$.

Find the Cartesian equation of the line.

(b) The lines $2x + 4y - 7 = 0$ and $3x - 5y + 8 = 0$ intersect at p.
Find the equation of the line through p which is parallel to the line $4x - 2y + 5 = 0$, without finding the co-ordinates of p.

(c) $a(-1, -5)$ is a point.
$c(3, -1)$ is the midpoint of $[ab]$.

(i) Find the co-ordinates of b.

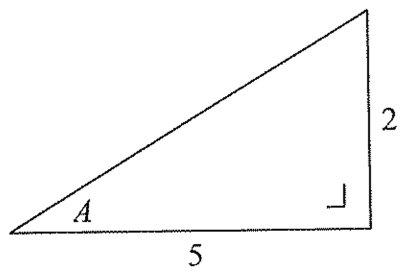
(ii) $cp \perp ab$, where p is the point (k^2, k).
Find the co-ordinates of the two points p_1 and p_2 which satisfy this condition.

(iii) Show that c divides $[p_1 p_2]$ in the ratio $2:1$.

4. (a) A is the angle shown in the diagram.

Find the value of $\sin 2A$, without using tables or a calculator.

Give your answer in the form $\frac{a}{b}$, where $a, b \in \mathbf{N}$.



(b) **(i)** Prove that $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$.

(ii) Given that $\cos A = \frac{7}{25}$, find the value of $\tan \frac{A}{2}$, $0 \le A \le 90°$.

(c) $[ab]$ is a diameter of the circle of centre o.

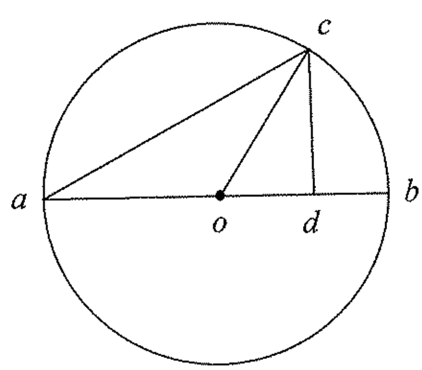
c is a point on the circle and $cd \perp ab$.

$|ab| = 26$ and area $\Delta aoc = 78$.

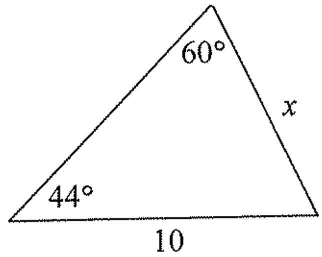
(i) Find $|\angle aoc|$, correct to one decimal place.

(ii) Find $|ac|$, correct to one decimal place.

(iii) Find $|ad|$, correct to the nearest integer.



5. (a) Find x, using the given diagram.
Give your answer correct to the nearest whole number.



(b) **(i)** Solve $2\sin^2 x + 4\cos^2 x = 3, \quad 0 \le x \le 2\pi$.

(ii) Prove that

$$\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$$

(c) A triangle has sides a, b and c. The angles opposite a, b and c are A, B and C, respectively.

(i) Prove that $a^2 = b^2 + c^2 - 2bc\cos A$.

(ii) Hence, prove that

$$\frac{\cos B}{\cos C} = \frac{c - b\cos A}{b - c\cos A}.$$

6. (a) Two dice are thrown. Find the probability that

(i) the same number is shown on each die

(ii) the sum of the numbers shown is less than 5.

(b) Solve the difference equation

$$2u_{n+2} = 3u_{n+1} - u_n, \quad n \ge 1$$

given that $u_1 = 1$ and $u_2 = 2$.

Find u_3 and u_4.

(c) To qualify for a competition, competitors must complete a race in a certain time interval. The probabilities that competitors A, B and C will finish the race within the time interval are $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{3}{4}$, respectively. All three competitors compete to qualify.

Find the probability that

(i) all three competitors qualify

(ii) only C qualifies

(iii) exactly two of the competitors qualify.

7. (a) The mean of the numbers $11,\ p,\ q,\ 27$ is $\bar{x}$.
Find, in terms of $\bar{x}$, the mean of the numbers $7,\ p-4,\ q-4,\ 23$.

(b) **(i)** How many three digit natural numbers can be formed using the digits 0 to 9 inclusive, if each digit is used only once?

What is the smallest number that can be formed?

(ii) Solve for n: $\dfrac{n!(n-1)!}{(n-2)!(n+1)!} = \dfrac{5}{7}$.

(c) There are x red discs and y black discs in a bag.
Two discs are taken at random from the bag.

(i) Find the probability that both discs are red.

(ii) Find the probability that one disc is red and one disc is black.

(iii) If there were 16 discs in the bag and the probability of getting 2 red discs was 0.375, find the number of red discs in the bag.

SECTION B

Answer ONE question from this section.

8. (a) Use integration by parts to find $\int_0^{\frac{\pi}{4}} x \sin 2x \, dx$.

(b) Use the ratio test to show that $\sum_{n=1}^{\infty} \frac{2+3^n}{4^n}$ is convergent.

(c) **(i)** Find the first three terms of the Maclaurin series for $\tan^{-1} x$.

(ii) Show that $\tan^{-1}\frac{3}{7} + \tan^{-1}\frac{2}{5} = \frac{\pi}{4}$.

(iii) Use your results from **(i)** and **(ii)** above to find an approximation for π.
Give your answer correct to four decimal places.

9. (a) Three numbers are picked at random from the digits 1 to 9.
Find the probability that 4 is the second smallest number picked.

(b) A factory fills bottles of water. The volume of water in each bottle is normally distributed with a mean of 500 ml and a standard deviation of 6 ml.
A bottle is selected at random.
Find the probability that the volume in the selected bottle is less than 480 ml.

(c) A company claims that a weedkiller it has developed will kill 42% of weeds within 12 hours of application.
50 out of 88 weeds treated responded within the 12 hour period.
At the 5% level of significance, is there evidence that the weedkiller is more effective in the treatment of weeds than the company claims?

10. (a) The set $\{1, 3, 5, 7\}$ forms a group under multiplication modulo 8.

(i) Write down the Cayley table for the group.

(ii) State the inverse of each element of the set.

(b) Q^+ is the set of positive rational numbers.

The operation $*$ is defined by $x * y = \frac{xy}{2}$.

(i) Show that the operation $*$ is associative.

(ii) Find the identity element.

(iii) Find the inverse of a.

(iv) Is $Q^+, *$ a group? Give reasons for your answer.

(c) Prove that in any group G if $g \in G$ then the set $H = \{g^n : n \in Z\}$ is a subgroup of G.

11. (a) Find the equation of the ellipse with foci $\left(\frac{-4}{3}, 0\right)$ and $\left(\frac{4}{3}, 0\right)$ and with eccentricity $\frac{4}{5}$.

(b) Show that under a transformation of the form

$$x' = ax + by + k_1$$
$$y' = cx + dy + k_2$$

(i) the midpoint of a line segment is mapped on to the midpoint of the image line segment

(ii) the image of the centroid of a triangle is the centroid of the image triangle.